Time series, Stationarity

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Econometrics

Nov. 14, 2019

1. Stationarity
2. Non stationarity
3. Dickey-Fuller test strategy

1. Stationarity
   - White noise
   - MA(q)
   - AR
2. Non stationarity
3. Dickey-Fuller test strategy

Definition: stationarity

X_t is said stationary if and only if

- $\mathbb{E}[X_t]$ and $Var(X_t)$ are constant.
- Covariance of X_t and X_{t-h} does not depend on t: $Cov(X_t, X_{t+h}) = \gamma(h)$.

This means that X_t "behavior" does not change in time.

White noise

The simplest time series model is the white noise:

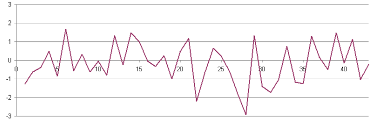
$$X_t = \varepsilon_t \text{ with } \varepsilon_t \text{ i.i.d.}$$

The past have no effect on present. The series has no memory.

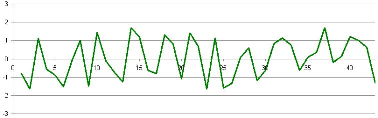
Theorem

White noise are stationary.

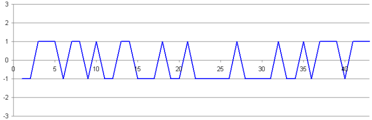
Gaussian



Uniform



Binary



MA(q) model

MA models some dependence: shocks perpetuate for a while

MA model

The series is a weighted mean of previous shocks.

$$X_t = \varepsilon_t + m_1\, \varepsilon_{t-1} + m_2\, \varepsilon_{t-2} + \cdots + m_q\, \varepsilon_{t-q}$$

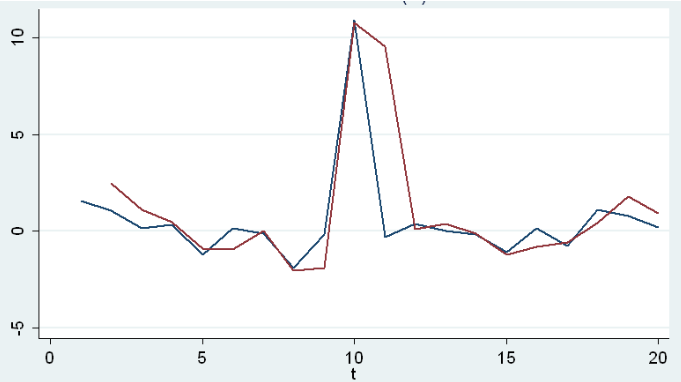
- $m_0 = 1$ by definition.
- q is the order of the MA: how long the shock effect lasts.

Theorem

MA(q) series are stationary.

MA(1)

$$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$$



White noise (blue) and MA(1) with $m_1 = 0.9$ (red)

Autocorrelation function

The autocorrelation is the covariance of X_t with its past, normalized by the variance:

$$ACF(k) = \frac{Cov(X_t, X_{t-k})}{Var(X_t)}$$

The ACF gives the order of the MA(q):

Theorem

If X_t is a $MA(q)$ then $ACF(q) \neq 0$ and for any $k > q$, $ACF(k) = 0$.

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

$$ACF(0) = \frac{Cov(X_t, X_t)}{Var(X_t)} = \frac{Var(X_t)}{Var(X_t)} = 1$$

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

$$ACF(0) = \frac{Cov(X_t, X_t)}{Var(X_t)} = \frac{Var(X_t)}{Var(X_t)} = 1$$

$$\begin{aligned}
ACF(1) &= \frac{Cov(X_t, X_{t-1})}{Var(X_t)} \\
&= \frac{Cov(\varepsilon_t + 0.9\varepsilon_{t-1}, \varepsilon_{t-1} + 0.9\varepsilon_{t-2})}{Var(\varepsilon_t + 0.9\varepsilon_{t-1})} \\
&= \frac{Cov(0.9\varepsilon_{t-1}, \varepsilon_{t-1})}{Var(\varepsilon_t) + 0.9^2\,Var(\varepsilon_{t-1})} = \frac{0.9}{1 + 0.9^2} = 0.55
\end{aligned}$$

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

$$ACF(0) = \frac{Cov(X_t, X_t)}{Var(X_t)} = \frac{Var(X_t)}{Var(X_t)} = 1$$

$$\begin{aligned} ACF(1) &= \frac{Cov(X_t, X_{t-1})}{Var(X_t)} \\ &= \frac{Cov(\varepsilon_t + 0.9\varepsilon_{t-1}, \varepsilon_{t-1} + 0.9\varepsilon_{t-2})}{Var(\varepsilon_t + 0.9\varepsilon_{t-1})} \\ &= \frac{Cov(0.9\varepsilon_{t-1}, \varepsilon_{t-1})}{Var(\varepsilon_t) + 0.9^2\, Var(\varepsilon_{t-1})} = \frac{0.9}{1 + 0.9^2} = 0.55 \end{aligned}$$

$$ACF(2) = \frac{Cov(X_t, X_{t-2})}{Var(X_t)} = \frac{0}{1 + 0.9^2} = 0$$

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

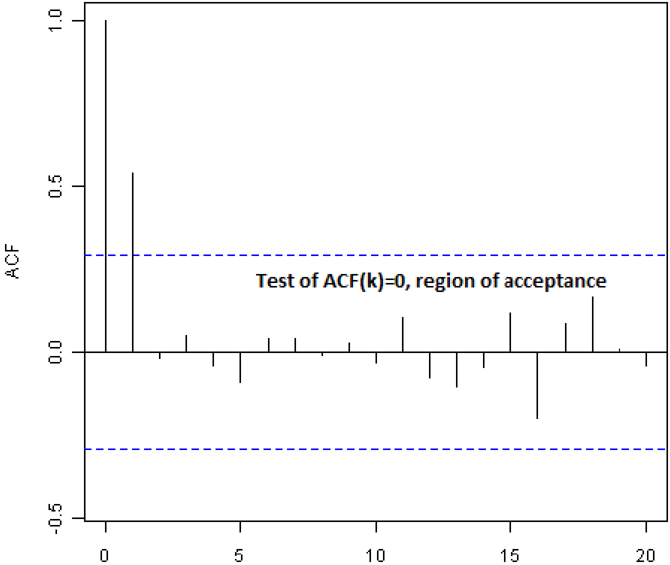
$$ACF(0) = \frac{Cov(X_t, X_t)}{Var(X_t)} = \frac{Var(X_t)}{Var(X_t)} = 1$$

$$\begin{aligned} ACF(1) &= \frac{Cov(X_t, X_{t-1})}{Var(X_t)} \\ &= \frac{Cov(\varepsilon_t + 0.9\varepsilon_{t-1}, \varepsilon_{t-1} + 0.9\varepsilon_{t-2})}{Var(\varepsilon_t + 0.9\varepsilon_{t-1})} \\ &= \frac{Cov(0.9\varepsilon_{t-1}, \varepsilon_{t-1})}{Var(\varepsilon_t) + 0.9^2\, Var(\varepsilon_{t-1})} = \frac{0.9}{1+0.9^2} = 0.55 \end{aligned}$$

$$ACF(2) = \frac{Cov(X_t, X_{t-2})}{Var(X_t)} = \frac{0}{1+0.9^2} = 0$$

$$ACF(3) = \frac{Cov(X_t, X_{t-3})}{Var(X_t)} = \frac{0}{1+0.9^2} = 0 \ \ldots$$

MA(1) autocorrelation function. $m_1 = 0.9$ then ACF(1)=0.55



The Lag operator: L

Let's L be the operator that moves the time index to the past of 1 unit: $L\varepsilon_t = \varepsilon_{t-1}$ and $LX_t = X_{t-1}$

$$X_t = \varepsilon_t + m_1\ \varepsilon_{t-1} + m_2\ \varepsilon_{t-2} + \cdots + m_q\ \varepsilon_{t-q}$$
$$X_t = \varepsilon_t + m_1\ L\varepsilon_t + m_2\ L^2\varepsilon_t + \cdots + m_q\ L^q\varepsilon_t$$
$$X_t = (1 + m_1 L + m_2 L^2 + \cdots + +m_q L^q)\varepsilon_t$$

The order of the polynomial in L is the same as the order of the MA.

AR(1) model

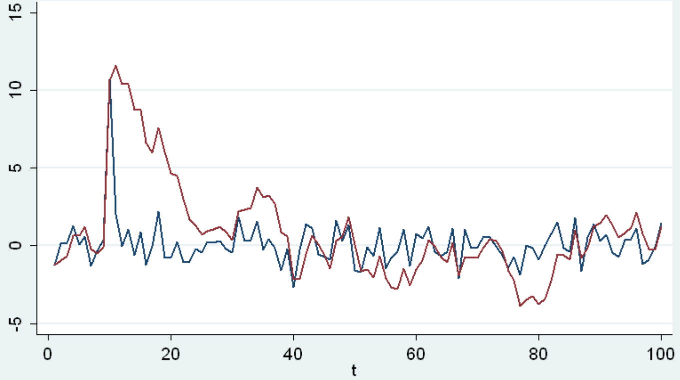
X_t is AutoRegressive of order 1, if it depends directly on its past.

AR(1) model

$$X_t = a_1 \; X_{t-1} + \varepsilon_t$$
$$\Leftrightarrow X_t - a_1 \; X_{t-1} = \varepsilon_t$$
$$\Leftrightarrow (1 - a_1 \; L) X_t = \varepsilon_t$$

AR(1)

$$X_t = 0.9\ X_{t-1} + \varepsilon_t$$

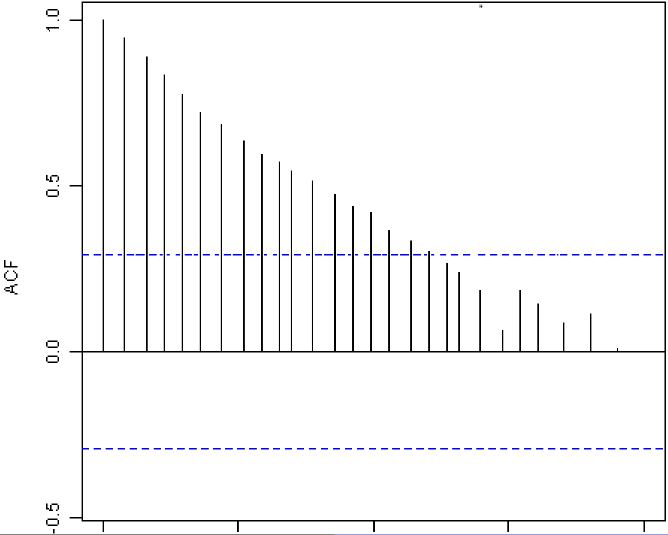


White noise (blue) and AR(1) with $a_1 = -0.9$ (red)

AR(1) are MA(∞)

$$
\begin{aligned}
X_t &= a_1 \; X_{t-1} + \varepsilon_t \\
(1 - a_1 \; L) X_t &= \varepsilon_t \\
X_t &= (1 - a_1 \; L)^{-1} \varepsilon_t \\
X_t &= (1 + a_1 \; L + a_1^2 L^2 + a_1^3 L^3 + \dots) \varepsilon_t \\
X_t &= \varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + a_1^3 \varepsilon_{t-3} + \dots
\end{aligned}
$$

The autocorrelation is no very informative on AR's:
$X_t = 0.9\ X_{t-1} + \varepsilon_t$



What if $a_1 = 1$?

The inversion $(1 - a_1\ L)^{-1}$ only works if $|a_1| < 1$
Note that $a_1 < 0$ is strange and $a_1 > 1$ is absurd, so $a_1 \in]0; 1]$.

Unit root

What if $a_1 = 1$? Then the binomial in L, $(1 - a_1\ L)$ has root $-1/a_1 = -1$. Non-stationary.

$$(1-L)X_t = \varepsilon_t \quad \Leftrightarrow X_t = X_{t-1} + \varepsilon_t \quad \Rightarrow Var(X_t) = Var(X_{t-1}) + \sigma^2_\varepsilon$$

$Var(X_t)$ can't be constant.

Theorem

If X_t is a AR(1), then the binomial $1 - a_1 L$ does not have 1 (nor -1) for root and X_t is stationary.

AR(p) model

In AutoRegressive models, X_t depends directly on its past.

AR model

$$X_t = a_1 \; X_{t-1} + a_2 \; X_{t-2} + \cdots + a_p \; X_{t-p} + \varepsilon_t$$

$$\Leftrightarrow (1 - a_1 \; L - a_2 \; L^2 + \cdots - a_p \; L^p) X_t = A(L) \; X_t = \varepsilon_t$$

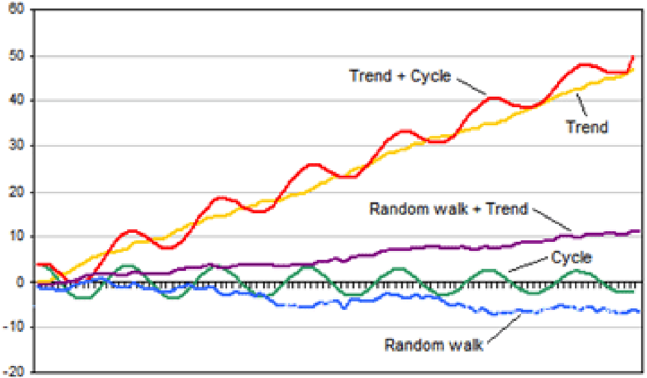
- $a_0 = 1$ by definition.
- p is the order of the AR: how long the past affects the present.
- the polynomial $A(x)$ has no unit root.

Theorem

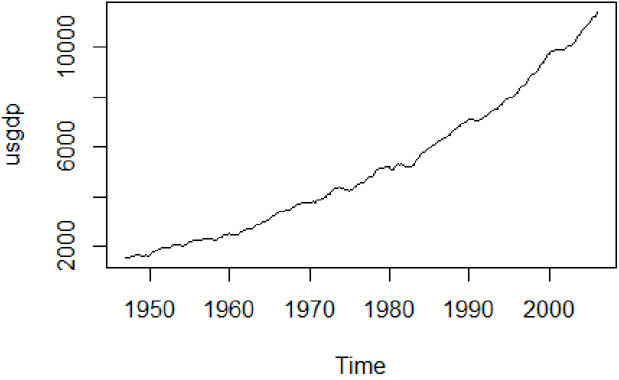
If X_t is a AR(p), then $A(x)$ has no unit root and X_t is stationary.

1. Stationarity
2. Non stationarity
    - GDP
    - Random walk?
3. Dickey-Fuller test strategy

Non stationarities:
seasonality, tend, random walk

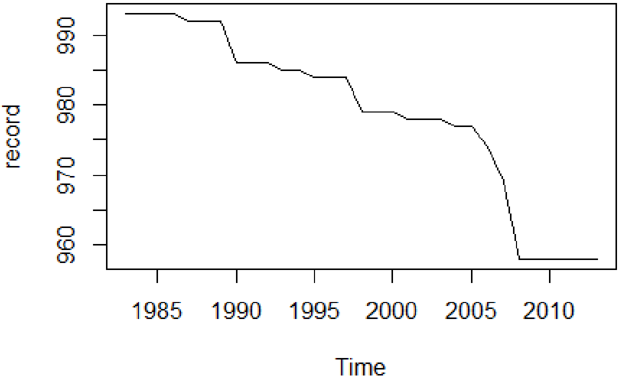


US GDP

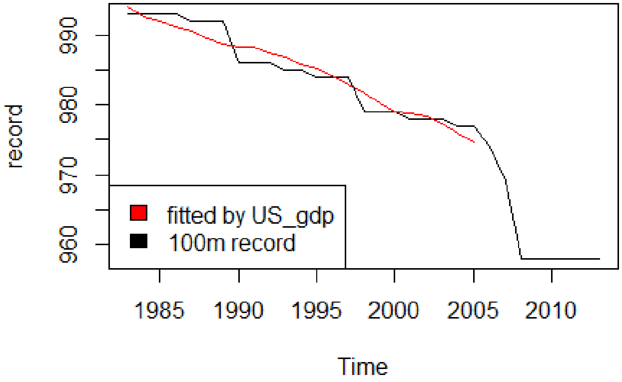


Clearly non stationary.

100m record

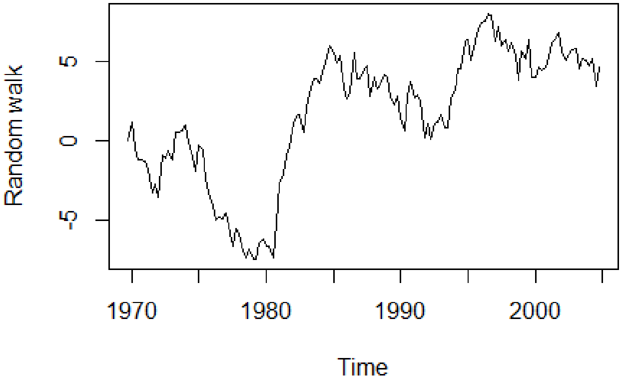


100m record

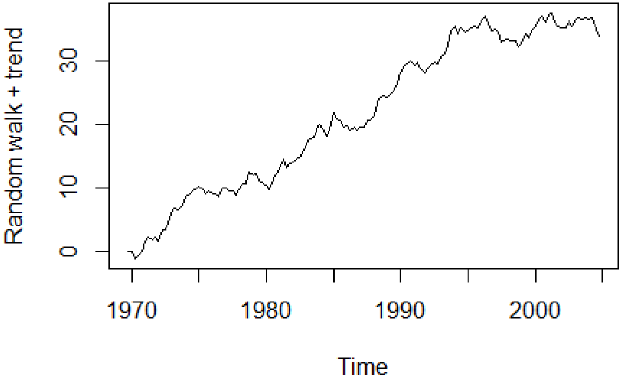

record
990
980
970
960
1985
1990
1995
2000
2005
2010
Time
fitted by US_gdp
100m record

```
Call:
lm(formula = record ~ year_gdp)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.012e+03  1.778e+00  569.40  < 2e-16 ***
year_gdp     -8.513e-04  5.379e-05  -15.83 3.81e-13 ***

Multiple R-squared:  0.9226,Adjusted R-squared:  0.919
```

Random walk



Random walk + trend



Random walk model

AR(1) model

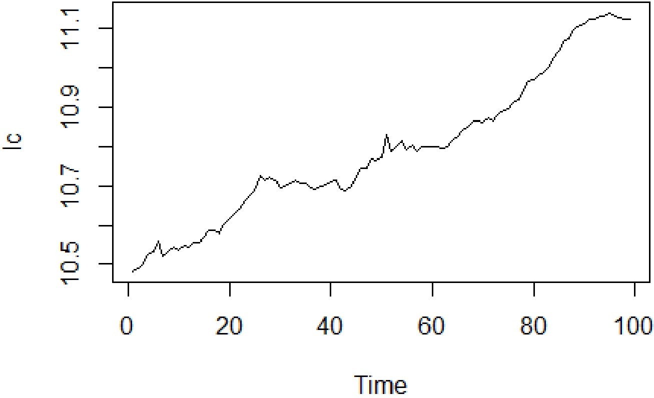
$$X_t = -a_1 \; X_{t-1} + \varepsilon_t$$

$$\Leftrightarrow (1 + a_1 \; L)X_t = \varepsilon_t$$

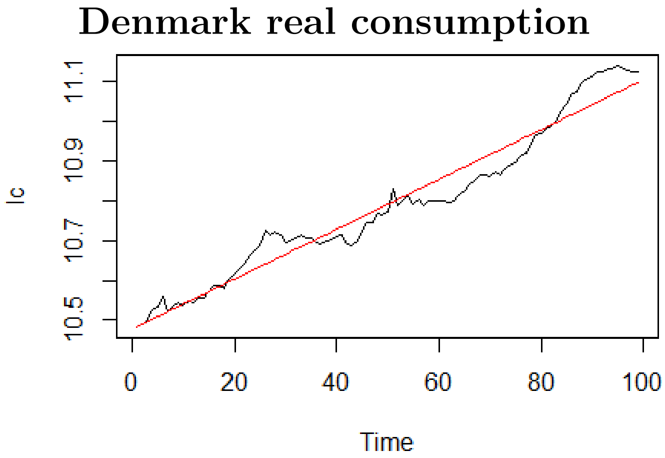
Unit root

$a_1 = -1 \; ! \quad \Rightarrow \quad L(X_t) = \varepsilon_t$

Denmark real consumption



Non stationary $\Rightarrow$ Random walk?



Non stationary ⇒ Trend?

Consider the model:

$$
\begin{aligned}
Z_t &= \alpha + \beta t - a_1 Z_{t-1} + \varepsilon_t \\
(1-L)Z_t &= \alpha + \beta t + (-a_1 - 1) Z_{t-1} + \varepsilon_t \\
(1-L)Z_t &= \alpha + \beta t + \rho Z_{t-1} + \varepsilon_t
\end{aligned}
$$

Looks like a linear model, but behaves very differently if $\rho = 0$ (“the null”).
And again differently if $\alpha \neq 0$ and/or $\beta \neq 0$
So we need a specific test (not a T-test) in each situation.

```
> library(urca)
> df=ur.df(X,type="trend")
> summary(df)

lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Estimate Std. Error t value  Pr(>|t|)
(Intercept)  0.3227245  0.1502083   2.149   0.0327 *
z.lag.1     -0.0329780  0.0166319  -1.983   0.0486 *
tt          -0.0004194  0.0009767  -0.429   0.6680
z.diff.lag  -0.0230547  0.0652767  -0.353   0.7243
```

Last column from linear model $\rightarrow$ false if unit-root.

$$(1-L)Z_t = \alpha + \beta t + \rho Z_{t-1} + \varepsilon_t$$

Output continues with adequate stats:

```
Value of test-statistic is: -1.9828 1.8771 2.7371
```

- t-value for $\rho = 0$
- Fisher stat for $(\rho, \alpha, \beta) = (0, 0, 0)$
- Fisher stat for $(\rho, \beta) = (0, 0)$

```
Critical values for test statistics:
      1pct  5pct  10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

$$(1-L)Z_t = \alpha + \beta t + \rho Z_{t-1} + \varepsilon_t$$

```
Value of test-statistic is: -1.9828 1.8771 2.7371
      1pct  5pct  10pct
tau3 -3.99 -3.43 -3.13
phi3  8.43  6.49  5.47
```

If the test stat for $\rho = 0$ is **larger** than tau3 then accept the unit-root. No absolute values here!
In this example, -1.98>-3.13, we accept $\rho = 0$ at 90%.

- If we accept $\rho = 0$, check that the full model is ok.
  - If the test stat [2.7] for (ρ, β) is larger than phi3 [5.47], then we reject $(\rho, \beta) = (0, 0)$, the full model is ok [No].
  - If not, then $\beta = 0$ and the full model is wrong. Move to model 2 (no trend).
- If we reject $\rho = 0$, it's a classical lineal model, check the trend with the first table.

```
> summary(ur.df(y=lc,type='drift')
lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)

Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.0038899  0.0841706   0.046    0.963
z.lag.1      0.0003199  0.0078044   0.041    0.967
z.diff.lag  -0.1240402  0.1028634  -1.206    0.231

Value of test-statistic is: 0.041 11.1569

Critical values for test statistics:
1pct  5pct 10pct
tau2 -3.51 -2.89 -2.58
phi1  6.70  4.71  3.86
```

$$(1-L)Z_t = \alpha + \rho Z_{t-1} + \varepsilon_t$$

```
Value of test-statistic is: 0.041 11.1569
      1pct  5pct  10pct
tau2 -3.51 -2.89 -2.58
phi1 6.70 4.71 3.86
```

If the test stat for $\rho = 0$ is **larger** than tau2 then accept the unit-root. No absolute values here!
In this example, 0.04>-2.58, we accept $\rho = 0$ at 90%.

- If we accept $\rho = 0$, check that model 2 is ok.
  - If the test stat [11] is larger than phi1 [6.7], then we reject $(\rho, \alpha) = (0, 0)$, model 2 is ok [even at 99%].
  - If not, then $\alpha = 0$ and model 2 is wrong. Move to model 1 (no drift)
- If we reject $\rho = 0$, it's a classical lineal model, check the drift with the first table.

```
> summary(ur.df(y=lc,type='none'))
lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)

Estimate Std. Error t value Pr(>|t|)
z.lag.1      0.0006805  0.0001433   4.749 7.24e-06 ***
z.diff.lag -0.1243891  0.1020458  -1.219    0.226

Value of test-statistic is: 4.7485

Critical values for test statistics:
1pct  5pct 10pct
tau1 -2.6 -1.95 -1.61
```

$$(1-L)Z_t = \rho Z_{t-1} + \varepsilon_t$$

Value of test-statistic is: 4.7485

```
       1pct  5pct  10pct
tau1 -2.6 -1.95 -1.61
```

If the test stat for $\rho = 0$ is **larger** than tau1 then accept the unit-root. No absolute values here!
In this example, 4.74>-1.6, we accept $\rho = 0$ at 90%, model 1 is ok.

All this can be done with more lags in the model (augmented DF model).
You can choose the lags:

```
summary(ur.df(y=lc,lags=3, type='trend'))
```

or leave it to R:

```
summary(ur.df(y=lc,type='trend',selectlags = "AIC"))
```

All the rest of the DF test strategy remains unchanged.