Time series, Stationarity

Hugo Harari-Kermadec

EPOG
Econometrics

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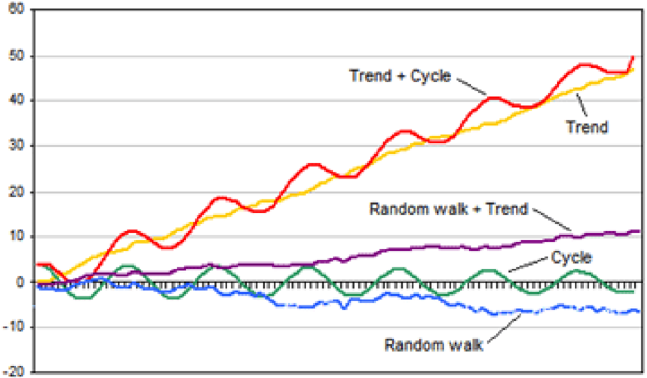
1. Stationarity
   - Alternatives
   - Definition
   - White noise
2. MA
3. AR

Seasonality, tend, random walk



Definition: stationarity

X_t is said stationary if and only if

- $\mathbb{E}[X_t]$ and $Var(X_t)$ are constant.
- Covariance of X_t and X_{t-h} does not depend on t: $Cov(X_t, X_{t+h}) = \gamma(h)$.

This means that X_t "behavior" does not change in time.

White noise

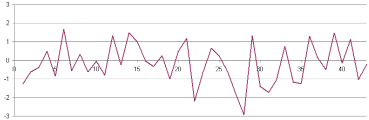
The simplest time series model is the white noise:

$$X_t = \varepsilon_t \text{ with } \varepsilon_t \text{ i.i.d.}$$

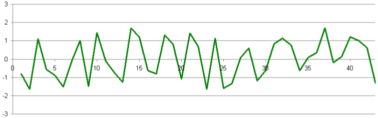
The past have no effect on present. The series has no memory.

Theorem

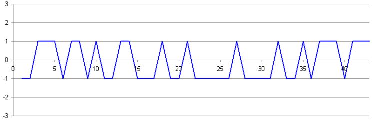
White noise are stationary.



Gaussian



Uniform



Binary

MA(q) model

MA models some dependence: shocks perpetuate for a while

MA model

The series is a weighted mean of previous shocks.

$$X_t = \varepsilon_t + m_1\, \varepsilon_{t-1} + m_2\, \varepsilon_{t-2} + \cdots + m_q\, \varepsilon_{t-q}$$

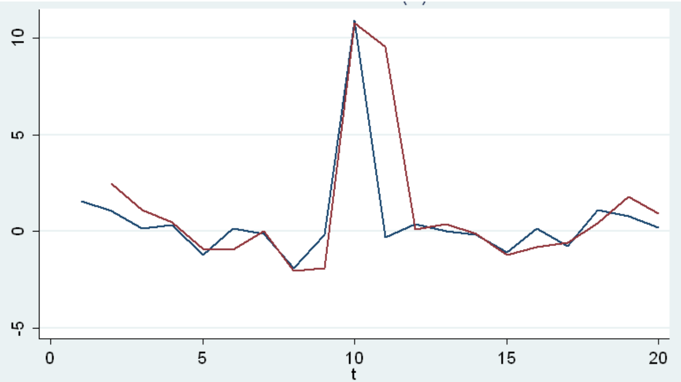
- $m_0 = 1$ by definition.
- q is the order of the MA: how long the shock effect lasts.

Theorem

MA(q) series are stationary.

MA(1)

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$



White noise (blue) and MA(1) with $m_1 = 0.9$ (red)

Autocorrelation function

The autocorrelation is the covariance of X_t with its past, normalized by the variance:

$$ACF(k) = \frac{Cov(X_t, X_{t-k})}{Var(X_t)}$$

The ACF gives the order of the MA(q):

Theorem

If X_t is a MA(q) then $ACF(q) \neq 0$ and for any $k > q$, $ACF(k) = 0$.

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

$$ACF(0) = \frac{Cov(X_t, X_t)}{Var(X_t)} = \frac{Var(X_t)}{Var(X_t)} = 1$$

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

$$ACF(0) = \frac{Cov(X_t, X_t)}{Var(X_t)} = \frac{Var(X_t)}{Var(X_t)} = 1$$

$$\begin{aligned} ACF(1) &= \frac{Cov(X_t, X_{t-1})}{Var(X_t)} \\ &= \frac{Cov(\varepsilon_t + 0.9\varepsilon_{t-1}, \varepsilon_{t-1} + 0.9\varepsilon_{t-2})}{Var(\varepsilon_t + 0.9\varepsilon_{t-1})} \\ &= \frac{Cov(0.9\varepsilon_{t-1}, \varepsilon_{t-1})}{Var(\varepsilon_t) + 0.9^2\, Var(\varepsilon_{t-1})} = \frac{0.9}{1 + 0.9^2} = 0.55 \end{aligned}$$

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$$ACF(2) = \frac{Cov(X_t, X_{t-2})}{Var(X_t)} = \frac{0}{1 + 0.9^2} = 0$$

$X_t = \varepsilon_t + 0.9\varepsilon_{t-1}$

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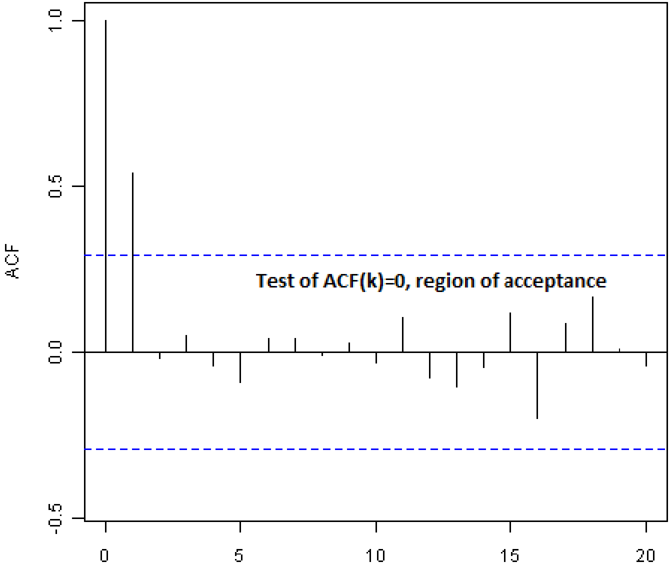
$$ACF(2) = \frac{Cov(X_t, X_{t-2})}{Var(X_t)} = \frac{0}{1+0.9^2} = 0$$
$$ACF(3) = \frac{Cov(X_t, X_{t-3})}{Var(X_t)} = \frac{0}{1+0.9^2} = 0 \ \ldots$$

MA(1) autocorrelation function. $m_1 = 0.9$ then ACF(1)=0.55



The Lag operator: L

Let's L be the operator that moves the time index to the past of 1 unit: $L\varepsilon_t = \varepsilon_{t-1}$ and $LX_t = X_{t-1}$

$$X_t = \varepsilon_t + m_1\ \varepsilon_{t-1} + m_2\ \varepsilon_{t-2} + \cdots + m_q\ \varepsilon_{t-q}$$
$$X_t = \varepsilon_t + m_1\ L\varepsilon_t + m_2\ L^2\varepsilon_t + \cdots + m_q\ L^q\varepsilon_t$$
$$X_t = (1 + m_1 L + m_2 L^2 + \cdots + +m_q L^q)\varepsilon_t$$

The order of the polynomial in L is the same as the order of the MA.

1. Stationarity
2. MA
3. AR
   - AR(1)
   - AR(p)
   - PACF

AR(1) model

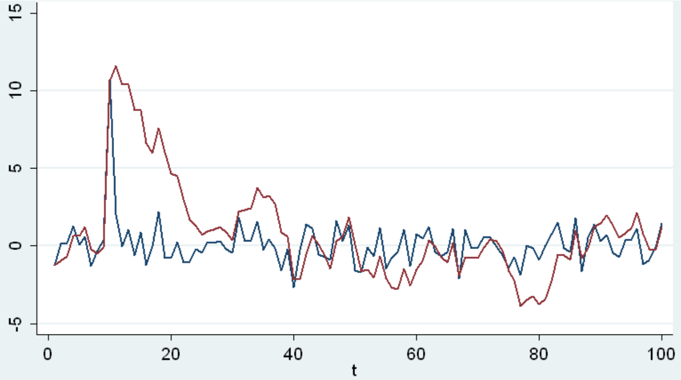
X_t is AutoRegressive of order 1, if it depends directly on its past.

AR(1) model

$$
\begin{aligned}
X_t &= a_1 \; X_{t-1} + \varepsilon_t \\
\Leftrightarrow X_t - a_1 \; X_{t-1} &= \varepsilon_t \\
\Leftrightarrow (1 - a_1 \; L) X_t &= \varepsilon_t
\end{aligned}
$$

AR(1)

$X_t = 0.9\ X_{t-1} + \varepsilon_t$



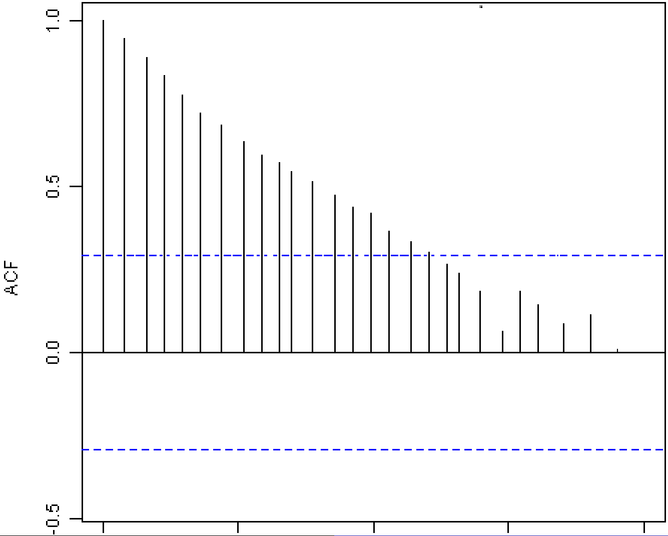
White noise (blue) and AR(1) with $a_1 = -0.9$ (red)

AR(1) are MA(∞)

$$
\begin{aligned}
X_t &= a_1\ X_{t-1} + \varepsilon_t \\
(1 - a_1\ L)X_t &= \varepsilon_t \\
X_t &= (1 - a_1\ L)^{-1}\varepsilon_t \\
X_t &= (1 + a_1\ L + a_1^2 L^2 + a_1^3 L^3 + \dots)\varepsilon_t \\
X_t &= \varepsilon_t + a_1\varepsilon_{t-1} + a_1^2\varepsilon_{t-2} + a_1^3\varepsilon_{t-3} + \dots
\end{aligned}
$$

The autocorrelation is no very informative on AR's:

$X_t = 0.9\ X_{t-1} + \varepsilon_t$



What if $a_1 = 1$?

The inversion $(1 - a_1\ L)^{-1}$ only works if $|a_1| < 1$
Note that $a_1 < 0$ is strange and $a_1 > 1$ is absurd, so $a_1 \in]0;1]$.

Unit root

What if $a_1 = 1$? Then the binomial in L, $(1 - a_1\ L)$ has root $-1/a_1 = -1$. Non-stationary.

$$(1-L)X_t = \varepsilon_t \quad \Leftrightarrow X_t = X_{t-1}+\varepsilon_t \quad \Rightarrow Var(X_t) = Var(X_{t-1})+\sigma_\varepsilon^2$$

$Var(X_t)$ can't be constant.

Theorem

If X_t is a AR(1), then the binomial $1 - a_1 L$ does not have 1 (nor -1) for root and X_t is stationary.

AR(p) model

In AutoRegressive models, X_t depends directly on its past.

AR model

$$X_t = a_1 \; X_{t-1} + a_2 \; X_{t-2} + \cdots + a_p \; X_{t-p} + \varepsilon_t$$

$$\Leftrightarrow (1 - a_1 \; L - a_2 \; L^2 + \cdots - a_p \; L^p) X_t = A(L) \; X_t = \varepsilon_t$$

- $a_0 = 1$ by definition.
- p is the order of the AR: how long the past affects the present.
- the polynomial $A(x)$ has no unit root.

Theorem

If X_t is a AR(p), then $A(x)$ has no unit root and X_t is stationary.

Partial autocorrelation function

PACF is much more informative:

$$X_t = a_1\ X_{t-1} + a_2\ X_{t-2} + \cdots + a_p\ X_{t-p} + \varepsilon_t$$

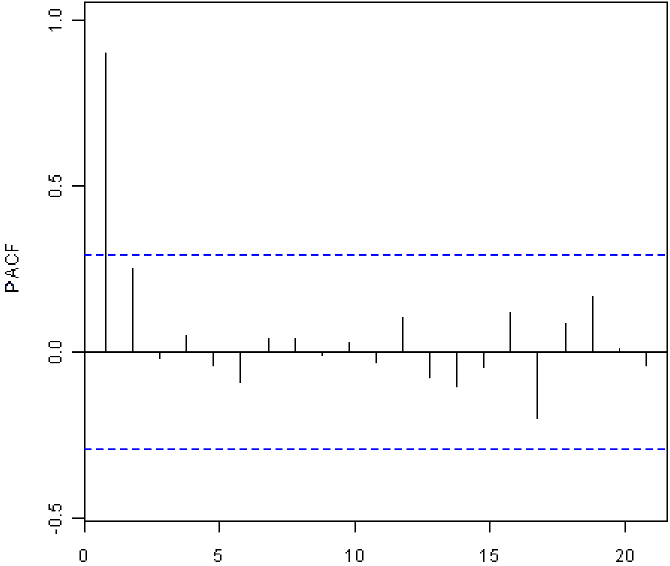
is a linear model with p explanatory variables.
PACF(k) is the coefficient of the last variable of the linear model explaining X_t by its k past values.

$$X_t = b_1\ X_{t-1} + b_2\ X_{t-2} + \cdots + PACF(k)\ X_{t-k} + \varepsilon_t$$

Theorem

If X_t is a AR(p) then $PACF(p) \neq 0$ and for any $k > p$, $PACF(k) = 0$.

The PACF gives the order of the AR. For $X_t = 0.9\ X_{t-1} + \varepsilon_t$



4 ARMA(p,q)

- Definition
- Orders
- Forecasting

ARMA(p,q) model

$$A(L)\ X_t = M(L)\ \varepsilon_t$$

$$X_t - a_1\ X_{t-1} + \cdots - a_p\ X_{t-p} = \varepsilon_t + m_1\ \varepsilon_{t-1} + \cdots + m_q\ \varepsilon_{t-q}$$

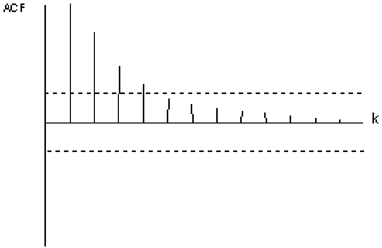
- $a_0 = m_0 = 1$ by definition.
- $A(x)$ has no unit root.
- $A(x)$ and $M(x)$ have no common root.

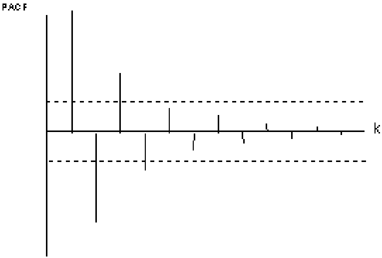
Theorem

If X_t is a ARMA(p,q), then $A(x)$ has no unit root and X_t is stationary.

Orders p and q

X_t has an AR part $\Rightarrow$ its ACF doesn't vanish.
X_t has a MA part $\Rightarrow$ its PACF doesn't vanish.
Both ACF and PACF fail !





Information criteria

We use an penalized adequation criterium instead.
Start with small p and q and

1. Estimate the parameters a_i and m_j.
2. Compute the log-likelihood (i.e. adequation)
3. Penalize by $(p+q)$ for AIC, $\log(T) * (p+q)$ for BIC
4. Increase p or q and start at stage 1.

The best model minimize the criterium.
The R function `auto.arima` does the job alone (package forecast).

```
 arima(data, order=c(1,0,1))
```
fits an ARMA(1,1)

```
> ARIMA(1,0,1)
  Coefficients:
       ar1      ma1
       0.7533   -0.7218
  s.e.  0.0457   0.1208
  sigma2 estimated as 230.4: log likelihood = -170.06
  AIC = 344.13 BIC = 347.56
```

`arima(data, order=c(2,0,0))` ⇒ `BIC = 350.64`
`arima(data, order=c(0,0,2))` ⇒ `BIC = 345.23`
`arima(data, order=c(2,0,1))` ⇒ `BIC = 354.21`
`arima(data, order=c(1,0,2))` ⇒ `BIC = 346.56`

The best model is a MA(2):

```
bestmodel <- arima(data, order=c(0,0,2))
```

To check the final model, we control that the residuals are a white noise. This portmanteau test check that there is no autocorrelation in the residuals.

```
>library(stats)
>Box.test(bestmodel$residuals, type= "Ljung-Box")

Box-Pierce test


data:  bestmodel$residuals
X-squared = 0.0032013, df = 1, p-value = 0.9549

acf(bestmodel$residuals)
```

Large p-value : accept the null, the residuals are iid.

Once you have chosen a model, you can **forecast**:

```
bestforecast<-forecast.Arima(bestmodel, h=3)
```

Point	Forecast	Lo 80	Hi 80	Lo 95	Hi 95
43	67.7	48.2	87.2	37.9	97.5
44	67.7	47.5	87.9	36.8	98.6
45	67.7	46.8	88.6	35.7	99.7